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[m1+; v 1.68; Prn:13/12/2006; 11:11] P.2(1-18) M. Belishev. V. Sharafutdinov / Bull. Sci. math. ••• (••••) •••-•••

1	In the scope of inverse problems of reconstructing a manifold from boundary measurements,	1
2	the following question is of great theoretical and applied interest: to what extent are the topol-	2
3	ogy and geometry of M determined by the DN map? It is proved in the two-dimensional case	3
4	that M is determined by Λ_{cl} up to a conformal equivalence [1,5]. There is the conjecture that Λ_{cl}	4
5	determines M up to an isometry in the case of $n \ge 3$. The latter is proved for real analytic man-	5
6	ifolds [6]. In the general case, it is proved that the boundary C^{∞} -jet of the metric is determined	6
7	by A_{\perp} for $n > 3$ [7]	7

by Λ_{cl} for $n \ge 3$ [7].

In [1], an explicit formula is obtained which expresses the Euler characteristic of Mthrough $\Lambda_{\rm cl}$ in the case of a two-dimensional M with a connected boundary. The Euler char-acteristic completely determines the topology of M in the latter case. In the three-dimensional case, the vector DN map $\overline{A}: C^{\infty}(T(\partial M)) \to C^{\infty}(T(\partial M))$ is defined on the space of vector fields in [2], and some formulas are obtained which express the Betti numbers $\beta_1(M)$ and $\beta_2(M)$ in terms of Λ_{cl} and $\hat{\Lambda}$.

Here we present a multidimensional generalization of the latter results. We define a DN map on the space of differential forms of all degrees and express Betti numbers in terms of the map. As well as in the case of n = 2, 3; the background of our formula is the Friedrichs decomposition of the space of harmonic fields. We also consider the Hilbert transform on differential forms and express it in terms of the DN map.

2. Preliminaries

Here, following [8] and mostly adhering to notations of this book, we recall some known facts on differential forms.

Let $\Omega^k(M)$ be the space of smooth real exterior differential forms of degree k and $\Omega(M) =$ $\bigoplus_{k=0}^{n} \Omega^{k}(M)$, the graded algebra of all forms. We use the following standard operators on $\Omega(M)$: the differential d, codifferential δ , Laplace operator $\Delta = d\delta + \delta d$, and Hodge star \star . Recall the relations

$$\star \star = (-1)^{k(n-k)}, \quad \star \delta = (-1)^k d\star, \quad \star d = (-1)^{k+1} \delta \star \quad \text{on} \quad \Omega^k(M).$$

The L^2 -product on $\Omega(M)$ is defined by $(\alpha, \beta) = \int_M \alpha \wedge \star \beta$ under the agreement that $\int_M \varphi = 0$ for $\varphi \in \Omega^k(M)$ with k < n. Recall Green's formula

$$(dlpha,eta)-(lpha,\deltaeta)=\int\limits_{\partial M}i^*(lpha\wedge\stareta),$$

where $i: \partial M \to M$ is the embedding. For $\alpha \in \Omega(M)$, the form $i^*\alpha$ will be sometimes called the boundary trace of α .

Elements of the space

$$\mathcal{H}^{k}(M) = \left\{ \lambda \in \Omega^{k}(M) \mid d\lambda = 0, \ \delta \lambda = 0 \right\}$$

are named harmonic fields. Recall the L^2 -orthogonal Hodge–Morrey decomposition

$$\Omega^{k}(M) = \mathcal{E}^{k}(M) \oplus \mathcal{C}^{k}(M) \oplus \mathcal{H}^{k}(M).$$

Here

and

$$\mathcal{C}^{k}(M) = \left\{ \delta \alpha \mid \alpha \in \Omega^{k+1}(M), \ i^{*}(\star \alpha) = 0 \right\}.$$

 $\mathcal{E}^{k}(M) = \left\{ d\alpha \mid \alpha \in \Omega^{k-1}(M), \ i^{*}\alpha = 0 \right\}$



There are two finite dimensional subspaces distinguished in $\mathcal{H}^k(M)$	
$\mathcal{H}^{k}_{r}(M) = \{ \lambda \in \mathcal{H}^{k}(M) \mid i^{*}\lambda = 0 \}$	
$\mathcal{H}_{D}^{k}(M) = \{\lambda \in \mathcal{H}^{k}(M) \mid i^{*}(z^{*}) = 0\}$	
$\mathcal{H}_N(M) = \{ \lambda \in \mathcal{H} \mid M \mid t \mid (\star \lambda) = 0 \}$	
whose elements are named Dirichlet and Neumann harmonic fields respective hese spaces are expressed by	ely. Dimensions of
$\dim \mathcal{H}_N^k(M) = \dim \mathcal{H}_D^{n-k}(M) = \beta_k(M),$	
where $\beta_k(M)$ is the <i>k</i> th Betti number. There are two L^2 -orthogonal Friedrich	s decompositions
$\mathcal{H}^k(M) = \mathcal{H}^k_D(M) \oplus \mathcal{H}^k_{co}(M), \qquad \mathcal{H}^k(M) = \mathcal{H}^k_N(M) \oplus \mathcal{H}^k_{ex}(M).$	
Here	
$\mathcal{H}^{k}(M) = \{\lambda \in \mathcal{H}^{k}(M) \mid \lambda = d\alpha\} \qquad \mathcal{H}^{k}(M) = \{\lambda \in \mathcal{H}^{k}(M) \mid \lambda = \delta\}$	Sal
$t_{ex}(M) = \{x \in \mathcal{H} \mid M\} \mid x = au\}, t_{co}(M) = \{x \in \mathcal{H} \mid M\} \mid x = 0$	^{/u}]
The operator \star maps the space $\mathcal{H}_D^k(M)$ isomorphically onto $\mathcal{H}_N^{n-k}(M)$. Ispaces	Introduce the trace
$i^*\mathcal{H}^k(M) = \left\{ i^*\lambda \mid \lambda \in \mathcal{H}^k(M) \right\}, \qquad i^*\mathcal{H}^k_N(M) = \left\{ i^*\lambda_N \mid \lambda_N \in \mathcal{H}^k_N(M) \right\}$	<i>I</i>)}.
A Neumann harmonic field λ_N is uniquely determined by its trace $i^*\lambda_N$. There of the space $i^*\mathcal{H}_N^k(M)$ is equal to $\beta_k(M)$. Let us prove the equality	fore the dimensior
$i^*\mathcal{H}^k(M) = \mathcal{E}^k(\partial M) + i^*\mathcal{H}^k_N(M).$	(2.1)
indeed, any harmonic field $\lambda \in \mathcal{H}^k(M)$ can be represented in the form	
$\lambda = d\eta + \lambda_N, \lambda_N \in \mathcal{H}^k_N(M)$	
by the second Friedrichs decomposition. This implies	
$i^*\lambda = i^*d\eta + i^*\lambda_N = d(i^*\eta) + i^*\lambda_N.$	
Conversely, for $\varphi \in \Omega^{k-1}(\partial M)$ and $\lambda_D \in \mathcal{H}_D^{k+1}(M)$,	
$\int d\varphi \wedge i^*(\star \lambda_D) = \int d(\varphi \wedge i^*(\star \lambda_D)) = 0.$	
$\partial M \qquad \partial M$	
Thus, the form $\alpha = d\varphi$ satisfies	
$d\alpha = 0$ and $\int \alpha \wedge i^*(\star \lambda_D) = 0 \forall \lambda_D \in \mathcal{H}_D^{k+1}(M).$	(2.2)
^{∂M} By Theorem 3.2.5 of [8], (2.2) is the necessary and sufficient condition for the	e existence of such
$\lambda \in \mathcal{H}^{k}(M)$ that $\alpha = i^{*}\lambda$. By the first of Friedrichs decompositions, a harmonic field $\lambda \in \mathcal{H}^{k}(M)$ can	be represented as
$\lambda = \delta lpha + \lambda_D, \lambda_D \in \mathcal{H}^k_D(M).$	(2.3)
We will need the following remark: in the representation, the form α can be $d\alpha = 0$ and $\Delta \alpha = 0$. Indeed, first consider representation (2.3) with some α by Hodge–Morrey	e chosen such that and decompose α

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This implies $\delta \alpha = \delta d\beta$. Therefore, in (2.3), α can be replaced with the form $\tilde{\alpha} = d\beta$ which satisfies $d\tilde{\alpha} = 0$. Next, if the form α in (2.3) satisfies $d\alpha = 0$, then it satisfies also the Eq. $\Delta \alpha = 0$ as is seen by applying the operator *d* to Eq. (2.3). A similar remark is valid on the second Friedrichs decomposition.

3. DN operator

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For any $0 \leq k \leq n - 1$, the DN operator	
$\Lambda: \Omega^k(\partial M) \to \Omega^{n-k-1}(\partial M)$	(3.1)

is defined as follows. Given $\varphi \in \Omega^k(\partial M)$, the boundary value problem

$$\begin{cases} \Delta \omega = 0, \\ i^* \omega = \varphi, \quad i^* (\delta \omega) = 0 \end{cases}$$
(3.2)

is solvable, see Lemma 3.4.7 of [8]. The solution $\omega \in \Omega^k(M)$ is unique up to an arbitrary Dirichlet harmonic field $\lambda_D \in \mathcal{H}^k_D(M)$. Therefore the form

$$\Lambda \varphi = j^* (\star d\omega) = (-1)^{k+1} j^* (\delta \star \omega) \tag{3.3}$$

is independent of the choice of the solution ω and Λ is a well defined operator.

In the scalar case of k = 0, our definition is equivalent to the classical one. Indeed, in this case the boundary value problem (3.2) coincides with (1.1) and definition (3.3) gives

$$\Lambda \varphi = \frac{\partial \omega}{\partial \nu} \mu_{\partial} \quad (\varphi \in \Omega^0(\partial M)),$$

where $\mu_{\partial} \in \Omega^{n-1}(\partial M)$ is the boundary volume form. Thus, in the case of k = 0, our operator Λ differs from the classical operator Λ_{cl} by the presence of the factor μ_{∂} . However, some authors prefer to consider the form-valued operator $\Lambda : \Omega^0(\partial M) \to \Omega^{n-1}(\partial M)$, see for example [9].

The boundary value problem (3.2) can be written in a slightly different form as the following statement shows.

Lemma 3.1. Given $\varphi \in \Omega^k(\partial M)$, let $\omega \in \Omega^k(M)$ be a solution to the boundary value problem (3.2). Then $d\omega \in \mathcal{H}^{k+1}(M)$ and $\delta\omega = 0$. In particular, (3.2) is equivalent to the boundary value problem

$$\begin{cases} \Delta \omega = 0, & \delta \omega = 0, \\ i^* \omega = \varphi. \end{cases}$$
(3.4)

Proof. Let $\lambda = d\omega \in \Omega^{k+1}(M)$. We state that λ is a harmonic field. Indeed, $d\lambda = dd\omega = 0$. Since d and Δ commute,

$$\Delta \lambda = \Delta d\omega = d\Delta \omega = 0.$$

The boundary conditions

$$i^*\delta\lambda = i^*\delta d\omega = -i^*d\delta\omega = -d(i^*\delta\omega) = 0$$

and

$$i^*(\star d\lambda) = i^*(\star dd\omega) = 0$$

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are esticled. Thus,) solves the hourdary value mechan	
are satisfied. Thus, λ solves the boundary value problem	
$\Delta \lambda = 0, i^*(\star d\lambda) = 0, i^* \delta \lambda = 0.$	
This implies, with the help of Proposition 3.4.5(iv) of [8], that λ is a harmonic	field.
Let now $\varepsilon = \delta \omega$. Then ε is a harmonic field. Indeed, $\delta \varepsilon = \delta \delta \omega = 0$ and $d\varepsilon = \delta \omega$.	$= d\delta\omega = -\delta d\omega =$
$-\delta \lambda = 0$. Since $i = 0$ by the second of the boundary conditions (3.2), ε is a L field. We have thus proved that $\varepsilon = \delta \omega$ is a co-exact harmonic field and it is a Γ	Dirichlet harmonic
field. This implies, by the first Friedrichs decomposition, that $\varepsilon = 0$. \Box	
The operator A is nonnegative in the following sense: the integral	
r the operator 24 is nonnegative in the following sense, the integral	
$arphi\wedge\Lambdaarphi$	
∂M	
is nonnegative for any $\varphi \in \Omega(\partial M)$. This follows from the next statement.	Given two forms
$\varphi, \psi \in \Omega(\partial M)$, let ω and ε be the corresponding solutions to the boundary value	lue problem (3.2)
$\begin{cases} \Delta \omega = 0, \\ i^* \omega = \omega i^* (\delta \omega) = 0 \end{cases} \qquad \begin{cases} \Delta \varepsilon = 0, \\ i^* \varepsilon = \psi i^* (\delta \varepsilon) = 0 \end{cases}$	(3.5)
$\left(i \omega = \varphi, i (0 \omega) = 0, (i 0 \varphi, (0 0) 0) \right)$	
r nen	
$\varphi \wedge \Lambda \psi = \int \psi \wedge \Lambda \varphi = (d\omega, d\varepsilon) + (\delta \omega, \delta \varepsilon).$	(3.6)
<i>әм әм</i>	
Indeed, by Green's formula	
$(d_{12}, d_{2}) = (a_{12}, b_{2}d_{2}) + \int (i^{*}a_{1}) \wedge (i^{*} + d_{2})$	
$(ub, uc) = (b, ouc) + \int_{\partial M} (t - b) \wedge (t - b) dc,$	
$(\delta\omega,\delta\varepsilon) = (\omega,d\delta\varepsilon) - \int (\iota^*\delta\varepsilon) \wedge (\iota^*\star\omega).$	
∂M	
To find the dual operator Λ^* , we write the first of equalities (3.6) in the form	n
$(\varphi + 2A)t = (1t + 2A\varphi)$	(3.7)
where $\star_2: \mathcal{Q}(\partial M) \to \mathcal{Q}(\partial M)$ is the Hodge star on ∂M . Setting $\psi = \star_2 \psi'$ on	(3.7) we obtain
$(\alpha + \alpha A + \alpha) h' = (+\alpha) h' + \alpha A (\alpha) = (h' - A (\alpha))$	(017),
$(\psi, \gamma_{0}) = (\gamma_{0} \psi, \gamma_{0}) = (\psi, \gamma_{0}) = (\psi, \gamma_{0})$. The last equality holds because $+\gamma$ is the L^{2} isometry of $O(\partial M)$. We have thu	sobtained
The fast equality holds because \star_{∂} is the <i>L</i> -isometry of 52 (0 <i>M</i>). We have the	
$\Lambda^{\star} = \star_{\partial} \Lambda \star_{\partial}$.	(3.8)
The kernel and range of the operator Λ are described by the following	
Lemma 3.2. The kernel of Λ coincides with the range of Λ . A form $\varphi \in \Omega^{I}$ Ker $\Lambda = \operatorname{Ran} \Lambda$ if and only if it is the trace of a harmonic field, i.e., $\varphi = i^* \lambda$ fo	$k^{k}(\partial M)$ belongs to $r \ \lambda \in \mathcal{H}^{k}(M).$
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Proof. We first prove the equality Ker $A = i^* \mathcal{H}(M)$, vhere $\mathcal{H}(M) = \bigoplus_{k=0}^n \mathcal{H}^k(M)$ is the space of all harmonic fields. If $\varphi \in \text{Ker } A$ and ω is a solution to be boundary value problem (3.2), then $\varphi = i^* \omega$ and $(d\omega, d\omega) + (\delta\omega, \delta\omega) = \int_{\partial M} \varphi \wedge A\varphi = 0$ by (3.6). This means that $\omega \in \mathcal{H}^k(M)$. Conversely, if $\varphi = i^* \omega$ for $\omega \in \mathcal{H}(M)$, then ω is a solution to the boundary value problem (3.2) and $A\varphi = i^*(\star d\omega) = 0$. Next, we prove the equality Ran $A = i^* \mathcal{H}(M)$. Let $\psi \in \text{Ran } A$, $\psi = A\varphi$. This means the existence of a solution $\omega \in \Omega(M)$ to the boundar alue problem $\left\{ \begin{aligned} \Delta \omega = 0, \delta \omega = 0, \\ i^* \omega = \varphi, i^*(\star d\omega) = \psi. \end{aligned} \right.$ By Lemma 3.1, $d\omega$ is a harmonic field. Therefore $\star d\omega$ is a harmonic field too. Hence $\psi = \frac{1}{*}(\star d\omega) \in i^* \mathcal{H}(M)$. Conversely, let $\psi \in i^* \mathcal{H}(M)$, i.e., $\psi = i^* \lambda, \lambda \in \mathcal{H}(M)$. (3.5) By the second Friedrichs decomposition, the harmonic field $\star \lambda$ can be represented as $\star \lambda = d\omega + \lambda_N$, (3.10) where λ_N is a Neumann harmonic field and ω is chosen such that (see the remark at the end of lection 2) $\delta \omega = 0, \Delta \omega = 0.$ This implies $A(i^*\omega) = i^*(\star d\omega).$ Applying the operator $i^* \star$ to (3.10), we obtain $i^*(\star d\omega) = \pm i^* \lambda.$ wo last equations imply $i^* \lambda = \pm A(i^* \omega).$ Comparing this equality with (3.9), we see that $\psi = A(\pm i^* \omega)$, i.e., $\psi \in \text{Ran } A. \square$ Corollary 3.3. The operator A possesses the following properties: A = 0, A = 0.	JID:BULSCI 6	AID:2223 /FLA [m1+; v 1.68; Prn:13/12/2006; 11:11] P.6 M. Belishev, V. Sharafutdinov / Bull. Sci. math. ••• (••••) •••-•••	(1-18)
Proof. We first prove the equality $\operatorname{Ker} A = i^* \mathcal{H}(M),$ where $\mathcal{H}(M) = \bigoplus_{i=0}^{n} \mathcal{H}^{k}(M)$ is the space of all harmonic fields. If $\varphi \in \operatorname{Ker} A$ and ω is a solution to the boundary value problem (3.2), then $\varphi = i^* \omega$ and $(d\omega, d\omega) + (\delta\omega, \delta\omega) = \int_{\partial M} \varphi \wedge A\varphi = 0$ by (3.6). This means that $\omega \in \mathcal{H}^{k}(M)$. Conversely, if $\varphi = i^* \omega$ for $\omega \in \mathcal{H}(M)$, then ω is a solution to the boundary value problem (3.2) and $A\varphi = i^*(\star d\omega) = 0$. Next, we prove the equality Ran $A = i^* \mathcal{H}(M)$. Let $\psi \in \operatorname{Ran} A$, $\psi = A\varphi$. This means the existence of a solution $\omega \in \Omega(M)$ to the boundar alue problem $\begin{cases} \Delta \omega = 0, \delta \omega = 0, \\ i^* \omega = \varphi, i^*(\star d\omega) = \psi. \end{cases}$ By Lemma 3.1, $d\omega$ is a harmonic field. Therefore $\star d\omega$ is a harmonic field too. Hence $\psi = i^*(\star d\omega) \in i^* \mathcal{H}(M)$. Conversely, let $\psi \in i^* \mathcal{H}(M)$, i.e., $\psi = i^* \lambda, \lambda \in \mathcal{H}(M).$ (3.5) By the second Friedrichs decomposition, the harmonic field $\star \lambda$ can be represented as $\star \lambda = d\omega + \lambda_N,$ (3.10) where λ_N is a Neumann harmonic field and ω is chosen such that (see the remark at the end of election 2) $\delta\omega = 0, \Delta\omega = 0.$ This implies $\Lambda(i^*\omega) = \pm i^* \lambda.$ wo last equations imply $i^* \lambda = \pm \Lambda(i^*\omega).$ Comparing this equality with (3.9), we see that $\psi = \Lambda(\pm i^*\omega)$, i.e., $\psi \in \operatorname{Ran} \Lambda. \square$ Corollary 3.3. The operator Λ possesses the following properties: $\Delta d = 0, \qquad \Delta A = 0.$	-		
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where $\mathcal{H}(M) = \bigoplus_{k=0}^{n} \mathcal{H}^{k}(M)$ is the space of all harmonic fields. If $\varphi \in \text{Ker } A$ and ω is a solutio o to boundary value problem (3.2), then $\varphi = i^{*}\omega$ and $(d\omega, d\omega) + (\delta\omega, \delta\omega) = \int_{\partial M} \varphi \wedge A\varphi = 0$ by (3.6). This means that $\omega \in \mathcal{H}^{k}(M)$. Conversely, if $\varphi = i^{*}\omega$ for $\omega \in \mathcal{H}(M)$, then ω is a solutio o to boundary value problem (3.2) and $A\varphi = i^{*}(\star d\omega) = 0$. Next, we prove the equality Ran $A = i^{*}\mathcal{H}(M)$. Let $\psi \in \text{Ran } A$, $\psi = A\varphi$. This means the existence of a solution $\omega \in \Omega(M)$ to the boundar alue problem $\begin{cases} \Delta\omega = 0, \delta\omega = 0, \\ i^{*}\omega = \varphi, i^{*}(\star d\omega) = \psi. \end{cases}$ By Lemma 3.1, $d\omega$ is a harmonic field. Therefore $\star d\omega$ is a harmonic field too. Hence $\psi = {}^{*}(\star d\omega) \in i^{*}\mathcal{H}(M)$. Conversely, let $\psi \in i^{*}\mathcal{H}(M)$, i.e., $\psi = i^{*}\lambda, \lambda \in \mathcal{H}(M)$. (3.9) By the second Friedrichs decomposition, the harmonic field $\star\lambda$ can be represented as $\star\lambda = d\omega + \lambda_N$, (3.10) where λ_N is a Neumann harmonic field and ω is chosen such that (see the remark at the end c fection 2) $\delta\omega = 0, \Delta\omega = 0.$ This implies $A(i^{*}\omega) = i^{*}(\star d\omega)$. Supplying the operator $i^{*} \star$ to (3.10), we obtain $i^{*}(\star d\omega) = \pm i^{*}\lambda$. Wo last equations imply $i^{*}\lambda = \pm A(i^{*}\omega)$. Comparing this equality with (3.9), we see that $\psi = A(\pm i^{*}\omega)$, i.e., $\psi \in \text{Ran } A$. \Box Corollary 3.3. The operator A possesses the following properties:	$\operatorname{Ker} \Lambda$	$=i^*\mathcal{H}(M),$	
The form $i = (0, i) = (0, i)$ is the space of an number fields $h \neq v$ for h and w is defined of definition $h \neq v$ for $h \neq v$ for h and w is definition of the boundary value problem (3.2), then $\varphi = i^* \omega$ and $(d\omega, d\omega) + (\delta\omega, \delta\omega) = \int_{\partial M} \varphi \wedge A\varphi = 0$ by (3.6). This means that $\omega \in \mathcal{H}^k(M)$. Conversely, if $\varphi = i^* \omega$ for $\omega \in \mathcal{H}(M)$, then ω is a solution to the boundary value problem (3.2) and $A\varphi = i^*(\star d\omega) = 0$. Next, we prove the equality Ran $A = i^*\mathcal{H}(M)$. Let $\psi \in \text{Ran } A$, $\psi = A\varphi$. This means the existence of a solution $\omega \in \Omega(M)$ to the boundar alue problem $\begin{cases} \Delta \omega = 0, \delta \omega = 0, \\ i^* \omega = \varphi, i^*(\star d\omega) = \psi. \end{cases}$ By Lemma 3.1, $d\omega$ is a harmonic field. Therefore $\star d\omega$ is a harmonic field too. Hence $\psi = i^*(\star d\omega) \in i^*\mathcal{H}(M)$. Conversely, let $\psi \in i^*\mathcal{H}(M)$, i.e., $\psi = i^*\lambda, \lambda \in \mathcal{H}(M)$. (3.9) By the second Friedrichs decomposition, the harmonic field $\star\lambda$ can be represented as $\star\lambda = d\omega + \lambda_N$, (3.10) where λ_N is a Neumann harmonic field and ω is chosen such that (see the remark at the end of Section 2) $\delta\omega = 0, \qquad \Delta\omega = 0.$ This implies $A(i^*\omega) = i^*(\star d\omega).$ Applying the operator $i^* \star$ to (3.10), we obtain $i^*(\star d\omega) = \pm i^*\lambda.$ Wo last equations imply $i^*\lambda = \pm A(i^*\omega).$ Comparing this equality with (3.9), we see that $\psi = A(\pm i^*\omega)$, i.e., $\psi \in \text{Ran } A. \square$ Corollary 3.3. The operator A possesses the following properties: $A = 0, \qquad i^2 = 0.$	where $\mathcal{H}(M)$	$= \bigoplus_{k=1}^{n} \mathcal{H}^{k}(M)$ is the space of all harmonic fields. If $\omega \in \text{Ker } A$ and ω is a so	olution
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$\begin{cases} i^* \omega = \varphi, & i^*(\star d\omega) = \psi. \\ \text{By Lemma 3.1, } d\omega \text{ is a harmonic field. Therefore } \star d\omega \text{ is a harmonic field too. Hence } \psi = \\ *(\star d\omega) \in i^* \mathcal{H}(M). \text{ Conversely, let } \psi \in i^* \mathcal{H}(M), \text{ i.e.,} \\ \psi = i^* \lambda, \lambda \in \mathcal{H}(M). \end{cases} $ (3.9) By the second Friedrichs decomposition, the harmonic field $\star \lambda$ can be represented as $\star \lambda = d\omega + \lambda_N, $ (3.10) where λ_N is a Neumann harmonic field and ω is chosen such that (see the remark at the end of Section 2) $\delta \omega = 0, \qquad \Delta \omega = 0.$ This implies $\Lambda(i^* \omega) = i^*(\star d\omega).$ Applying the operator $i^* \star$ to (3.10), we obtain $i^*(\star d\omega) = \pm i^* \lambda.$ Wo last equations imply $i^* \lambda = \pm \Lambda(i^* \omega).$ Comparing this equality with (3.9), we see that $\psi = \Lambda(\pm i^* \omega)$, i.e., $\psi \in \text{Ran } \Lambda. \square$	[Δω	$=0, \delta\omega=0,$	
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Sy the second Friedrich's decomposition, the narmonic field $\star \lambda$ can be represented as $\star \lambda = d\omega + \lambda_N$, (3.10) where λ_N is a Neumann harmonic field and ω is chosen such that (see the remark at the end of Section 2) $\delta \omega = 0$, $\Delta \omega = 0$. This implies $\Lambda(i^*\omega) = i^*(\star d\omega)$. Applying the operator $i^* \star$ to (3.10), we obtain $i^*(\star d\omega) = \pm i^* \lambda$. Swo last equations imply $i^* \lambda = \pm \Lambda(i^*\omega)$. Comparing this equality with (3.9), we see that $\psi = \Lambda(\pm i^*\omega)$, i.e., $\psi \in \operatorname{Ran} \Lambda$. \Box Corollary 3.3. The operator Λ possesses the following properties: $\Lambda d = 0$, $d\Lambda = 0$, $\Lambda^2 = 0$.	$\psi = \iota$	$d \sum_{i=1}^{n} d_{i} = b_{i} d_{i} = b_{i} d_{i} + b_{i} d_{i} = b_{i} d_{i} d_{i} + b_{i} d_{i} = b_{i} d_{i} d_{i} + b_{i} d_{i} d_{i} = b_{i} d_{i} d_{i} + b_{i} d_{i} d_{i} = b_{i} d_{i} d_{i} d_{i} d_{i} + b_{i} d_{i} d_{i} d_{i} d_{i} d_{i} = b_{i} d_{i} $	(3.7)
* $\lambda = d\omega + \lambda_N$, (3.10 where λ_N is a Neumann harmonic field and ω is chosen such that (see the remark at the end of Section 2) $\delta \omega = 0$, $\Delta \omega = 0$. This implies $\Lambda(i^*\omega) = i^*(\star d\omega)$. Applying the operator $i^* \star$ to (3.10), we obtain $i^*(\star d\omega) = \pm i^* \lambda$. We last equations imply $i^*\lambda = \pm \Lambda(i^*\omega)$. Comparing this equality with (3.9), we see that $\psi = \Lambda(\pm i^*\omega)$, i.e., $\psi \in \operatorname{Ran} \Lambda$. \Box Corollary 3.3. The operator Λ possesses the following properties: $\Lambda d = 0$, $d\Lambda = 0$, $\Lambda^2 = 0$.	by the secon	a Friedrich's decomposition, the narmonic field */ can be represented as	
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This implies $\Lambda(i^*\omega) = i^*(\star d\omega).$ Applying the operator $i^*\star$ to (3.10), we obtain $i^*(\star d\omega) = \pm i^*\lambda.$ We olast equations imply $i^*\lambda = \pm \Lambda(i^*\omega).$ Comparing this equality with (3.9), we see that $\psi = \Lambda(\pm i^*\omega)$, i.e., $\psi \in \operatorname{Ran} \Lambda.$ \Box Corollary 3.3. The operator Λ possesses the following properties: $\Lambda d = 0$ $d\Lambda = 0$ $\Lambda^2 = 0$ (2.11)	$\delta \omega = 0$	$\Delta \omega = 0.$	
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Corollary 3.3. The operator Λ possesses the following properties:	Comparing t	his equality with (3.9), we see that $\psi = \Lambda(\pm i^*\omega)$, i.e., $\psi \in \operatorname{Ran} \Lambda$. \Box	
$Ad = 0 \qquad dA = 0 \qquad A^2 = 0 \qquad (2.11)$	Corollary 3	3 The operator A possesses the following properties:	
$a = 0$ $a = 0$ $a^2 = 0$ (4.1)	2010nary 3	$2 - 1 = 0$ $4^2 = 0$	(2 1 1)
$Aa = 0, \qquad aA = 0, \qquad A = 0. \tag{3.11}$	$\Lambda d =$	$a \Lambda = 0, \qquad \Lambda^2 = 0.$	(3.11)
Proof. The first of equalities (3.11) means that any exact form is the trace of a harmonic field	Proof. The	first of equalities (3.11) means that any exact form is the trace of a harmonia	c field.
This is true by (2.1) . The second of equalities (3.11) is equivalent to the obvious fact: the trace of	This is true b	y (2.1) . The second of equalities (3.11) is equivalent to the obvious fact: the tr	race of
harmonic field is a closed form. The last of equalities (3.11) follows from the relation Ker Λ =	a harmonic f	eld is a closed form. The last of equalities (3.11) follows from the relation Ke	$\operatorname{er} \Lambda =$

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[m1+: v 1.68: Prn:13/12/2006: 11:11] P.7(1-18) M. Belishev, V. Sharafutdinov / Bull. Sci. math. ••• (••••) •••-•••

Corollary 3.4. The operator $d \Lambda^{-1}$ is well defined on boundary traces of harmonic fields, i.e., the

equation $\Lambda \varphi = \psi$ has a solution φ for any $\psi \in i^* \mathcal{H}(M)$, and $d\varphi$ is uniquely determined by ψ .

In particular, the operator $d\Lambda^{-1}d: \Omega(\partial M) \to \Omega(\partial M)$ is well defined.

Proof. The boundary trace $\psi \in i^* \mathcal{H}(M)$ of a harmonic field belongs to Ran A by Lemma 3.2, so the equation $\Lambda \varphi = \psi$ is solvable. If $\Lambda \varphi_1 = \Lambda \varphi_2$, then the form $\varphi_1 - \varphi_2$ is closed since it is the trace of a harmonic field. Therefore $d\varphi_1 = d\varphi_2$. An exact form is the trace of a harmonic field by (2.1). □ **Remark 1.** There is some freedom in the definition of the DN operator. One can define the DN map as $\tilde{\Lambda} = (-1)^{kn} \star_{\partial} \Lambda : \Omega^k(\partial M) \to \Omega^k(\partial M).$ This operator preserves the degree of a form. Moreover, it is a nonnegative self-dual operator, i.e., $(\tilde{A}\varphi, \varphi) \ge 0$ and $\tilde{A}^* = \tilde{A}$ as is seen from (3.6) and (3.8). Thus, the operator \tilde{A} has more conventional properties than Λ . Just $\tilde{\Lambda}$ is used in [2]. Nevertheless, we have chosen Λ in our definition of the DN map since we share the opinion by J. Sylvester [9]: the DN operator should transform a k-form to an (n-k-1)-form. The operators A and A are equivalent in the following sense: given the Riemannian manifold ∂M , we can express $\tilde{\Lambda}$ through Λ and vise versa. **Remark 2.** Quite different definition of the DN map is chosen in [4]. By this definition, the DN operator maps a form $\varphi \in \Omega^k(M)|_{\partial M}$ to $\partial \omega / \partial \nu$, where ω is the solution to the boundary value problem (1.1). The main result of [4] is that the full symbol of the latter DN map $\Omega^k(M)|_{\partial M} \rightarrow$ $\Omega^k(M)|_{\partial M}$ determines the boundary C^{∞} -jet of the metric for any k. 4. Betti numbers If we know the kernel of Λ , we can write down some low bounds for Betti numbers as is seen from the following **Theorem 4.1.** Let Λ_k be the restriction of the operator Λ to $\Omega^k(\partial M)$. The kernel Ker Λ_k con-tains the space $\mathcal{E}^k(\partial M)$ of exact forms and dim [Ker $\Lambda_k / \mathcal{E}^k(\partial M)$] $\leq \min \{\beta_k(M), \beta_k(\partial M)\}.$ **Proof.** Consider the Hodge decomposition for ∂M $\Omega^{k}(\partial M) = \mathcal{C}^{k}(\partial M) \oplus \mathcal{E}^{k}(\partial M) \oplus \mathcal{H}^{k}(\partial M).$ The space of closed forms coincides with the sum of two last summands of the decomposition. The kernel Ker A_k consists of closed forms by Lemma 3.2 and contains all exact forms by (3.11), i.e., $\mathcal{E}^k(\partial M) \subset \operatorname{Ker} \Lambda_k \subset \mathcal{E}^k(\partial M) \oplus \mathcal{H}^k(\partial M).$ This implies $\dim \left[\operatorname{Ker} \Lambda_k / \mathcal{E}^k(\partial M)\right] \leq \dim \mathcal{H}^k(\partial M) = \beta_k(\partial M).$ By Lemma 3.2 and (2.1), $\operatorname{Ker} \Lambda_k = \mathcal{E}^k(\partial M) + j^* \mathcal{H}^k_N(M).$ Please cite this article as: M. Belishev, V. Sharafutdinov, Dirichlet to Neumann operator on differential forms, Bull. Sci. math. (2007), doi:10.1016/j.bulsci.2006.11.003

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Therefore	
$\dim \left[\operatorname{Ker} \Lambda_k / \mathcal{E}^k(\partial M) \right] \leqslant \dim \mathcal{H}_N^k(M) = \beta_k(M). \qquad \Box$	
The main result of the article is the following	
Theorem 4.2. For any $0 \le k \le n - 1$, the range of the operator	
$\Lambda + (-1)^{kn+k+n} d\Lambda^{-1} d: \Omega^k(\partial M) \to \Omega^{n-k-1}(\partial M)$	
is $i^*\mathcal{H}_N^{n-k-1}(M)$ and	
$\dim \operatorname{Ran} \left[\Lambda + (-1)^{kn+k+n} d \Lambda^{-1} d \right] = \beta_{n-k-1}(M).$	
Proof. We have to prove the equality	
$\left(\Lambda + (-1)^{kn+k+n} d\Lambda^{-1} d\right) \Omega^k(\partial M) = i^* \mathcal{H}_N^{n-k-1}(M).$	(4.1
Given $\varphi \in \Omega^k(\partial M)$, let $\omega \in \Omega^k(M)$ be a solution to the boundary value p Lemma 3.1, $d\omega \in \mathcal{H}^{k+1}(M)$. We apply the first Friedrichs decomposition to $d\omega$	problem (3.2). By ω
$d\omega = \delta \alpha + \lambda_D$, where $\lambda_D \in \mathcal{H}_D^{k+1}(M)$.	(4.2)
As is mentioned at the end of Section 2, the form $\alpha \in \Omega^{k+2}(M)$ can be chosen	such that
$d\alpha = 0, \qquad \Delta \alpha = 0.$	(4.3
We set $\beta = \star \alpha \in \Omega^{n-k-2}(M)$. (4.3) implies	
$\delta \beta = 0, \qquad \Delta \beta = 0.$	(4.4
Substituting the value $\alpha = (-1)^{k(n-k)} \star \beta$ into (4.2), we have	
$d\omega = (-1)^{k(n-k)}\delta \star \beta + \lambda_D.$	(4.5
Apply the operator i^* to Eq. (4.5)	
$i^*(d\omega) = (-1)^{k(n-k)} i^*(\delta \star \beta).$	(4.6
Using the relations	
$i^*(d\omega) = d(i^*\omega) = d\varphi$	
and	
$\delta \star \beta = (-1)^{n-k-1} \star d\beta,$	
we rewrite (4.6) in the form	
$d\varphi = (-1)^{kn+n+1}i^*(\star d\beta).$	(4.7
Formulas (4.4) and (4.7) mean that	
$d\varphi = (-1)^{kn+n+1} \Lambda i^* \beta.$	(4.8
Next, we apply the operator \star to Eq. (4.5)	
$\frac{1}{k(n-k)}$	

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and take the restriction to the boundary	
$i^*(\star d\omega) = (-1)^{k(n-k)}i^*(\star\delta\star\beta) + i^*(\star\lambda_D).$	(4.9)
The left-hand side of this formula is equal to $\Lambda \varphi$. Using the relation	
$\star\delta\star\beta = (-1)^{kn}d\beta,$	
we transform the first term on the right-hand side of (4.9) as follows:	
$i^*(\star\delta\star\beta) = (-1)^{kn}i^*d\beta = (-1)^{kn}d(i^*\beta).$	
Thus, (4.9) is equivalent to the equation	
$\Lambda \varphi = (-1)^k d(i^*\beta) + i^*(\star \lambda_D).$	(4.10)
The form $i^*\beta$ can be eliminated from the system of Eqs. (4.8) and (4.10). Indeed, (4.8) with the help of Corollary 3.3	implies
$d(i^*\beta) = (-1)^{kn+n+1} (d\Lambda^{-1}d)\varphi.$	
Inserting this expression into (4.10), we obtain	
$\left(\Lambda + (-1)^{kn+k+n} d\Lambda^{-1} d\right) \varphi = i^* (\star \lambda_D).$	
We have thus proved that the left-hand side of (4.1) is a subset of the right-hand side. To prove the converse inclusion, we first recall that	
$\mathcal{H}_D^k(M) \cap \mathcal{H}_N^k(M) = 0.$	
Together with Friedrichs decompositions, this implies that	
$\mathcal{H}^k(M) = \mathcal{H}^k_{\mathrm{ex}}(M) + \mathcal{H}^k_{\mathrm{co}}(M),$	
i.e., a harmonic field can be represented as a sum of exact and co-exact harmonic fields. Given $\lambda_N \in \mathcal{H}_N^{n-k-1}(M)$, the representation	
$\lambda_N = dlpha + \deltaeta$	(4.11)
exists by the remark of the previous paragraph. The forms α and β can be chosen such the	nat
$\delta \alpha = 0, \qquad \Delta \alpha = 0$ and	(4.12)
$d\beta = 0, \qquad \Delta\beta = 0.$	(4.13)
The latter statement is proved by the same argument as one used at the end of Section 2. We set	
$\omega = (-1)^{kn+1} \star \beta, \qquad \varepsilon = (-1)^{kn+k+n} \alpha.$	
Relations (4.12)–(4.13) imply	
$\delta \omega = 0, \Delta \omega = 0,$	(4.14)
$\delta arepsilon = 0, \Delta arepsilon = 0,$	(4.15)
and Eq. (4.11) is rewritten in the form	
$\lambda_N = \star d\omega + (-1)^{kn+k+n} d\varepsilon.$	(4.16)

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Applying the operator \star to the latter equation	we obtain
$+\lambda x = (-1)^{kn+k+n} (+ds - dw)$	(4.17)
$\mathbf{x}_{N} = (-1) \qquad (\mathbf{x}_{u} \mathbf{z} - u \mathbf{w}).$	(4.17)
we define forms $\varphi, \psi \in \Omega(\partial M)$ by	
$\varphi = \iota^* \omega, \qquad \psi = \iota^* \varepsilon.$	(4.18)
Restricting Eq. (4.16) to the boundary, we obt	ain
$i^*\lambda_N = i^*(\star d\omega) + (-1)^{\kappa n + \kappa + n} d(i^*\varepsilon).$	(4.19)
Eqs. (4.14) and the first of equalities (4.18) r rewritten as	nean that $i^*(\star d\omega) = \Lambda \varphi$. Therefore (4.19) can be
$i^*\lambda_N = \Lambda \varphi + (-1)^{kn+k+n} d(i^*\varepsilon).$	(4.20)
On the other hand, restricting Eq. (4.17) to the	e boundary, we obtain
$i^*(\star d\varepsilon) = d(i^*\omega).$	(4.21)
Eqs. (4.15) and the second of equalities (4.18) rewritten as	mean that $i^*(\star d\varepsilon) = \Lambda \psi$. Therefore (4.21) can be
$\Lambda \psi = d\varphi.$	(4.22)
Finally, we eliminate the form ψ from the Corollary 3.3 and obtain	system of Eqs. (4.20) and (4.22) with the help of
$i^*\lambda_N = (\Lambda + (-1)^{kn+k+n} d\Lambda^{-1} d)\varphi.$	
5. Hilbert transform	
One of equivalent definitions of the classic is as follows. Let $f = \varepsilon + i\omega$ be a holomorph and ε are conjugate by Cauchy–Riemann: $d\omega$ traces, then $T \frac{d\varphi}{d\theta} = \frac{d\psi}{d\theta}$. Returning to the general case, we define the	al Hilbert transform <i>T</i> on the unit circle $S = \{e^{i\theta}\}$ this function in the disc $\{re^{i\theta} \mid 0 \le r \le 1\}$ so that ω $\phi = \star d\varepsilon$. If $\varphi = \omega _S$ and $\psi = \varepsilon _S$ are the boundary the Hilbert transform as follows:
$T = d\Lambda^{-1} : i^* \mathcal{H}^k(M) \to i^* \mathcal{H}^{n-k}(M).$	
This is a well defined operator by Corollary forms and maps such forms again to exact for	3.4. In particular, T is defined on exact boundary ms, i.e.,
$T: \mathcal{E}^k(\partial M) \to \mathcal{E}^{n-k}(\partial M).$	
In the present section, we use <i>T</i> as the operator Let $\omega \in \Omega^k(M)$ and $\varepsilon \in \Omega^{n-k-2}(M)$ (0 \leq	or on the space of exact boundary forms. $k \leq n-2$) be two co-closed forms,
$\delta \omega = 0, \qquad \delta \varepsilon = 0.$	
The form ε is named the conjugate form of ω	if
$d\omega = \star d\varepsilon.$	(5.1)

 $\Delta \omega = 0,$ $\Delta \varepsilon = 0,$

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and $(-1)^{kn+k+n+1}\omega$ is the conjugate form of ε .	
Not any ω satisfying $\Delta \omega = 0$ and $\delta \omega = 0$ has a conjugate form. The remar	kable fact is that
the existence of the conjugate form can be checked in terms of the trace φ =	$= i^* \omega$ and of the
operator A.	
Theorem 5.1. A form $\omega \in \Omega^{\kappa}(M)$ satisfying $\Delta \omega = 0$ and $\delta \omega = 0$ has a conjutional only if the trace $\varphi = i^* \omega$ satisfies	gate form if and
$\left(\Lambda + (-1)^{kn+k+n} d\Lambda^{-1} d\right)\varphi = 0.$	(5.2)
In this case, if ε is the conjugate form of ω and $\psi = i^* \varepsilon$, then	
$Td\varphi = d\psi.$	(5.3)
Proof. <i>Necessity.</i> Let a co-closed form $\omega \in \Omega^k(M)$ have a conjugate co- $\Omega^{n-k-2}(M)$. Set $\varphi = i^* \omega$ and $\psi = i^* \varepsilon$. The forms ω and ε solve boundary value. Therefore	closed form $\varepsilon \in$ e problems (3.5)
$\Lambda \varphi = i^*(\star d\omega), \qquad \Lambda \psi = i^*(\star d\varepsilon).$	(5.4)
The second of equalities (5.4) and (5.1) imply	
$\Lambda \psi = i^* d\omega = d(i^* \omega) = d\varphi.$	
Applying the operator \star to (5.1), we get	
$\star d\omega = (-1)^{kn+k+n+1} d\varepsilon.$	
Together with the latter relation, the first of equalities (5.4) gives	
$\Lambda \varphi = (-1)^{kn+k+n+1} i^* d\varepsilon = (-1)^{kn+k+n+1} d(i^*\varepsilon) = (-1)^{kn+k+n+1} d\psi.$	
We have thus proved that	
$\int A\varphi = (-1)^{kn+k+n+1}d\psi,$	(5.5)
$\begin{cases} \Lambda \psi = d\varphi. \end{cases}$	(5.5)
Eliminating ψ from the latter system, we obtain (5.2). The second of equations (2.2)	5.5) is equivalen
to (5.3). Sufficiency Let a form $a \in O^{k}(\partial M)$ satisfy (5.2) and () has a solution to the	houndamy yolu
problem (3.2). Applying the operator $T = dA^{-1}$ to Eq. (5.2), we obtain	t boundary value
$\left(I + (-1)^{kn+k+n}T^2\right)d\varphi = 0,$	(5.6)
where I is the identity operator.	
By Corollary 3.4, the equation	
$\Lambda \psi = d\varphi$	(5.7)
is solvable. Fix a solution ψ to the equation and consider the boundary value pr	oblem
$\int d\varepsilon = \chi := (-1)^{kn+k+n+1} \star d\omega, \delta \varepsilon = 0,$	(5 8)
$i^*\varepsilon = \psi.$	(3.8)
By Theorem 3.2.5 of [8], the necessary and sufficient conditions for solvability are	y of the problem

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and	
$(\mathbf{x}, \lambda_{\mathcal{D}}) = \int \eta (\mathbf{x} \wedge i^* (\mathbf{x}, \mathbf{D})) \forall \lambda_{\mathcal{D}} \in \mathcal{H}^{n-k-1}(M)$	(5.9)
$(\chi, \kappa_D) = \int_{-\infty}^{\infty} \psi \wedge \iota (\kappa \kappa_D) \forall \kappa_D \in \mathcal{H}_D (M).$	(3.7)
The first condition is satisfied since	
$d_{11} = d_{12} d_{12} = d_{12} d_{13} d_{$	
$a\chi = \pm a \star a\omega = \pm \star \delta a\omega = \pm \star \Delta \omega = 0.$	
The second condition holds since, by (5.2),	
$i^*\chi = (-1)^{kn+k+n+1}i^*(\star d\omega) = (-1)^{kn+k+n+1}\Lambda\varphi = d\Lambda^{-1}d\varphi = d\psi.$	
It remains to check (5.9).	
The left-hand side of (5.9) is equal to zero for any Dirichlet harmonic field	λ_D . Indeed, sub-
stituting the value of χ from (3.8), we can write	
$(\chi, \lambda_D) = \pm (\star d\omega, \lambda_D) = \pm (d\omega, \star \lambda_D).$	
The right-hand side of the latter formula is zero by the second Friedrichs decom	position since $d\alpha$
is an exact harmonic field and $\star \lambda_D$ is a Neumann harmonic field. Condition (5.	(9) is thus reduced
r	
$\psi \wedge i^* \lambda_N = 0 \forall \lambda_N \in \mathcal{H}_N^{k+1}(M).$	(5.10)
∂M	
By Theorem 4.2, $i^* \mathcal{H}_N^{k+1}(M)$ coincides with the range of the operator	
$G = \Lambda + (-1)^{kn+k+n} d\Lambda^{-1} d: \Omega^{n-k-2}(\partial M) \to \Omega^{k+1}(\partial M).$	(5.11)
Therefore condition (5.10) can be rewritten as follows:	
$(\psi, \star_{\partial} G\eta) = \pm \int \psi \wedge G\eta = 0 \forall \eta \in \Omega^{n-k-2}(\partial M).$	
∂M	
In other words, ψ must belong to the kernel of the operator $(\star_{\partial} G)^*$,	
$(\star_{\partial} G)^* \psi = 0.$	(5.12)
One easily obtains from (3.8)	
$G^* = \star_\partial G \star_\partial$	
and	
$(+,C)^* = ++,C$	
$(\mathbf{x}_{\partial}\mathbf{O}) = \pm \mathbf{x}_{\partial}\mathbf{O}.$	
Therefore (5.12) is equivalent to the equation	
$G\psi = 0.$	(5.13)
Finally, substituting the values $\psi = \Lambda^{-1} d\varphi$ and	
$G = \Lambda + (-1)^{kn+k+n} d\Lambda^{-1} d$	
into (5.13) , we see that the latter equation is equivalent to (5.6) .	
We have thus proved the solvability of the boundary value problem (5.8).	The solution ε to
the problem is the conjugate form of ω . \Box	

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where the operator $G: \Omega(\partial M) \to \Omega(\partial M)$ is defined by (5.11). The Hilbert transform T maps $\mathcal{B}^k(\partial M)$ isomorphically onto $\mathcal{B}^{n-k}(\partial M)$. If $\tilde{T}_k: \mathcal{B}^k(\partial M) \to \mathcal{B}^{n-k}(\partial M)$ is the restriction of T to $\mathcal{B}^k(\partial M)$, then $\tilde{T}_k^{-1} = (-1)^{kn+k} \tilde{T}_{n-k}$. **Proof.** For a form $\varphi \in \Omega^{k-1}(\partial M)$ satisfying $G\varphi = 0$, let $\omega \in \Omega^{k-1}(M)$ be a solution to the boundary value problem (3.2). Then ω has a conjugate form $\varepsilon \in \Omega^{n-k-1}(M)$ by Theorem 5.1. The trace $\psi = i^* \varepsilon$ satisfies (5.3). The form ε has the conjugate $(-1)^{kn+k} \omega$, and $G \psi = 0$ by Theorem 5.1. Now, (5.3) shows that $Td\varphi = d\psi \in \mathcal{B}^{n-k}(\partial M)$. We have thus proved that T maps $\mathcal{B}^k(\partial M)$ to $\mathcal{B}^{n-k}(\partial M)$ and operator (5.14) is well defined. By (5.6), $\tilde{T}_{n-k}\tilde{T}_k = (-1)^{kn+k}I$. Therefore \tilde{T}_k is an isomorphism. \Box **Corollary 5.3.** Let $1 \le k \le n-1$. If $\beta_k(M) = \beta_{n-k}(M) = 0$, then the Hilbert transform T maps $\mathcal{E}^{k}(\partial M)$ isomorphically onto $\mathcal{E}^{n-k}(\partial M)$. If $T_k: \mathcal{E}^k(\partial M) \to \mathcal{E}^{n-k}(\partial M)$ is the restriction of T to $\mathcal{E}^k(\partial M)$, then $T_k^{-1} = (-1)^{kn+k} T_{n-k}$. **Proof.** If $\beta_k(M) = \beta_{n-k}(M) = 0$, then the operator G vanishes on $\Omega^{k-1}(\partial M)$ and on $\Omega^{n-k-1}(\partial M)$ by Theorem 4.2. Therefore $\mathcal{B}^k(\partial M) = \mathcal{E}^k(\partial M)$ and $\mathcal{B}^{n-k}(\partial M) = \mathcal{E}^{n-k}(\partial M)$. It remains to apply Corollary 5.2. \Box 6. DN map on highest degree forms

We prove here that the volume of the manifold can be easily determined from the DN map known on forms of highest degree.

Theorem 6.1. For $\varphi \in \Omega^{n-1}(\partial M)$, where $n = \dim M$, the function $\Lambda \varphi \in \Omega^0(\partial M)$ is constant and

$$\Lambda \varphi = \frac{1}{\operatorname{Vol}(M)} \int_{\partial M} \varphi.$$

This is a generalization of the classical formula

$$2\operatorname{Vol}(M) = \int_{\partial M} (x\,dy - y\,dx)$$

for a plane domain M.

Proof. Let ω be a solution to the boundary value problem (3.2). By Lemma 3.1, $d\omega$ is a harmonic field. The space $\mathcal{H}^n(M)$ of harmonic fields of highest degree consists of forms $C\mu$, where C =const and μ is the volume form. Thus,

$$d\omega = C\mu, \quad C = \text{const}$$

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 $\mathcal{B}^{k}(\partial M) = \left\{ d\varphi \mid \varphi \in \Omega^{k-1}(\partial M), \, G\varphi = 0 \right\} \subset \mathcal{E}^{k}(\partial M).$

Corollary 5.2. For $1 \le k \le n - 1$, introduce the space

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From this, σ	$\star d\omega = C$ and = $i^*(\star d\omega) = C$.	(6.1)

Write down the Stockes formula for ω

$$\int_{M} d\omega = \int_{\partial M} i^* \omega.$$

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Since $d\omega = C\mu$ and $i^*\omega = \varphi$, this gives

$$C\operatorname{Vol}(M) = \int_{\partial M} \varphi.$$

Together with (6.1), the last formula gives the statement of the theorem. \Box

Let μ_{∂} be the volume form of ∂M . Choosing a function $\lambda \in C^{\infty}(\partial M)$ and setting $\varphi = \lambda \mu_{\partial}$ in Theorem 6.1, we obtain

$$\operatorname{Vol}(M) = \frac{1}{\Lambda(\lambda\mu_{\partial})} \int\limits_{\partial M} \lambda\mu_{\partial}.$$

Perhaps, this formula is of some interest for applications, e.g., in electro impedance tomography. The question is whether the value of the constant function $\Lambda(\lambda\mu_{\partial})$ can be extracted from boundary measurements. If so, the volume Vol(*M*) can be determined from boundary measurements implemented on an arbitrarily small part of the boundary. Indeed, the function λ can be chosen to be supported in an arbitrary open subset of ∂M and it is enough to measure the value of the constant function $\Lambda(\lambda\mu_{\partial})$ at one point.

7. Recovering the additive real cohomology structure from the DN map

The exact cohomology sequence of the pair $(M, \partial M)$ looks as follows

$$\cdots \xrightarrow{\partial^*} H^k(M, \partial M) \xrightarrow{j^*} H^k(M) \xrightarrow{i^*} H^k(\partial M) \xrightarrow{\partial^*} H^{k+1}(M, \partial M) \xrightarrow{j^*} \cdots$$
(7.1)

We consider cohomologies with real coefficients. Recall that the finite dimensional vector spaces $H^k(M)$ are defined as cohomologies of the De Rham complex

$$\cdots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \cdots$$

Similarly, $H^{k}(\partial M)$ are cohomologies of the De Rham complex of the boundary. The operator i^{*} on (7.1) is defined by the equality $i^{*}[\omega]_{M} = [i^{*}\omega]_{\partial M}$, where $[\omega]_{M}$ is the cohomology class of a closed form ω in $H^{k}(M)$ and $[i^{*}\omega]_{\partial M}$ is the cohomology class of the form $i^{*}\omega$ in $H^{k}(\partial M)$. The definition is correct since d and i^{*} commute.

Let us recall the definition of relative cohomologies. Let $\Omega^k(M, \partial M)$ be the space of forms $\omega \in \Omega^k(M)$ satisfying $i^*\omega = 0$. If $i^*\omega = 0$ then also $i^*(d\omega) = 0$. Therefore we have the welldefined cochain complex

$$\cdots \xrightarrow{d} \Omega^{k-1}(M, \partial M) \xrightarrow{d} \Omega^k(M, \partial M) \xrightarrow{d} \Omega^{k+1}(M, \partial M) \xrightarrow{d} \cdots$$

The spaces $H^k(M, \partial M)$ are cohomologies of the latter complex. The operator j^* on (7.1) is induced by the embedding of pairs $j : (M, \emptyset) \subset (M, \partial M)$. In other words, $j^*[\omega]_{(M,\partial M)} = [\omega]_M$ induced by the embedding of pairs $j : (M, \emptyset) \subset (M, \partial M)$.

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for a closed form $\omega \in \Omega^k(M)$ satisfying $i^*\omega = 0$. The definition is correct since $[\omega]_M = 0$ if $[\omega]_{(M,\partial M)} = 0$. Finally, the coboundary operator ∂^* on (7.1) is defined as follows. Given a closed boundary form $\omega \in \Omega^k(\partial M)$, let $\alpha \in \Omega^k(M)$ be an extension of ω to M, i.e., $i^*\alpha = \omega$. The form $d\alpha$ is closed and has the zero boundary trace. We set $\partial^*[\omega]_{\partial M} = [d\alpha]_{(M \ \partial M)}$. One can check the correctness of the definition. Sequence (7.1) is exact, i.e., the kernel of each operator of the sequence coincides with the range of the preceding operator. This is the standard fact of cohomology theory [3].

Now, we pose the inverse problem: Given the data $(\partial M, \Lambda)$, one has to recover sequence (7.1) up to an isomorphism, i.e., to construct a sequence

$$\cdots \xrightarrow{\tilde{\partial}^*} \tilde{H}^k(M, \partial M) \xrightarrow{\tilde{j}^*} \tilde{H}^k(M) \xrightarrow{\tilde{i}^*} H^k(\partial M) \xrightarrow{\tilde{\partial}^*} \tilde{H}^{k+1}(M, \partial M) \xrightarrow{\tilde{j}^*} \cdots$$
(7.2)

of vector spaces and operators which is isomorphic to sequence (7.1). The latter means the exis-tence of a commutative diagram

$$\cdots \xrightarrow{\tilde{\partial}^*} \tilde{H}^k(M, \partial M) \xrightarrow{\tilde{j}^*} \tilde{H}^k(M) \xrightarrow{\tilde{i}^*} H^k(\partial M) \xrightarrow{\tilde{\partial}^*} \tilde{H}^{k+1}(M, \partial M) \xrightarrow{\tilde{i}^*} \cdots$$

$$(7.2)$$

$$\dots \xrightarrow{\partial^*} H^k(M, \partial M) \xrightarrow{j^*} H^k(M) \xrightarrow{i^*} H^k(\partial M) \xrightarrow{\partial^*} H^{k+1}(M, \partial M) \xrightarrow{j^*} \dots$$

where λ , μ are isomorphisms and ι is the identity operator.

We present the solution of the inverse problem based on the results of previous sections.

By Theorem 4.2, we can determine the spaces $i^*\mathcal{H}_N^k(M)$ from our data $(\partial M, \Lambda)$. We define

$$\tilde{H}^k(M) = i^* \mathcal{H}^k_N(M), \qquad \tilde{H}^k(M, \partial M) = i^* \mathcal{H}^{n-k}_N(M).$$

The homomorphism $\tilde{H}^k(M) \xrightarrow{\tilde{i}^*} H^k(\partial M)$ is defined as follows. If $\varphi = i^* \omega \in \tilde{H}^k(M)$ for $\omega \in$ $\mathcal{H}_{N}^{k}(M)$, then the form $\varphi \in \Omega^{k}(\partial M)$ is closed, and we set $\tilde{i}^{*}\varphi = [\varphi]_{\partial M}$.

In Section 5, we defined the Hilbert transform as an operator on the space of boundary traces of harmonic fields:

$$T = d\Lambda^{-1} : i^* \mathcal{H}^k(M) \to i^* \mathcal{H}^{n-k}(M).$$

Moreover, the following holds:

Lemma 7.1. The Hilbert transform maps traces of Neumann harmonic fields again to traces of Neumann harmonic fields, i.e.,

$$T = d\Lambda^{-1} : i^* \mathcal{H}^k_N(M) \to i^* \mathcal{H}^{n-k}_N(M).$$

Proof. Let $\psi \in i^* \mathcal{H}^k_N(M)$. By Theorem 4.2, ψ can be represented as

$$\psi = \left(\Lambda + (-1)^{kn+k+1} d\Lambda^{-1} d\right) \varphi$$

with some $\varphi \in \Omega^{n-k-1}(\partial M)$. From this

$$T\psi = d\Lambda^{-1}\psi = d\Lambda^{-1} \left(\Lambda + (-1)^{kn+k+1} d\Lambda^{-1} d\right)\varphi$$

$$= \left(d + (-1)^{kn+k+1} d\Lambda^{-1} d\Lambda^{-1} d\right)\varphi = \left(\Lambda + (-1)^{kn+k+1} d\Lambda^{-1} d\right)\Lambda^{-1} d\varphi.$$

The right-hand side of the latter formula belongs to $i^*\mathcal{H}_N^{n-k}(M)$ by the same Theorem 4.2. \Box

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We continue constructing sequence (7.2). On using Lemma 7.1, we define the homomorphism $\tilde{j}^*: \tilde{H}^k(M, \partial M) \longrightarrow \tilde{H}^k(M)$ as

$$\tilde{j}^* = (-1)^{kn+k+1}T : \tilde{H}^k(M, \partial M) = i^* \mathcal{H}_N^{n-k}(M) \to i^* \mathcal{H}_N^k(M) = \tilde{H}^k(M).$$

Finally, the homomorphism $\tilde{\partial}^*: H^k(\partial M) \to \tilde{H}^{k+1}(M, \partial M)$ is defined as

$$\tilde{\partial}^* = (-1)^{kn+k+n+1} \Lambda : H^k(\partial M) \longrightarrow i^* \mathcal{H}_N^{n-k-1}(M) = \tilde{H}^{k+1}(M, \partial M).$$

More precisely, we observe that, given a closed form $\varphi \in \Omega^k(\partial M)$, the form $\Lambda \varphi$ belongs to the space $i^*\mathcal{H}_N^{n-k-1}(M)$ in view of Theorem 4.2 and of the equality $\Lambda \varphi = (\Lambda + (-1)^{kn+k+n} d\Lambda^{-1} d)\varphi$. We set

$$\tilde{\partial}^* [\varphi]_{\partial M} = (-1)^{kn+k+n+1} \Lambda \varphi.$$

The definition is correct since Ad = 0.

We have thus constructed sequence (7.2). Next, we will define the vertical isomorphisms λ and μ participating on diagram (7.3).

The operator $\tilde{H}^k(M) \xrightarrow{\lambda} H^k(M)$ is defined as follows. If $\varphi = i^* \omega$ for $\omega \in \mathcal{H}^k_N(M)$, then $\lambda \varphi = [\omega]_M$. It is the isomorphism because there exists a unique Neumann harmonic field in any cohomology class.

The homomorphism $\tilde{H}^{k}(M, \partial M) \xrightarrow{\mu} H^{k}(M, \partial M)$ is defined as follows. If $\varphi = i^{*}\omega$ for $\omega \in \mathcal{H}_{N}^{n-k}(M)$, then the form $\star \omega \in \mathcal{H}_{D}^{k}(M)$ is closed and $i^{*}(\star \omega) = 0$. We set $\mu \varphi = [\star \omega]_{(M,\partial M)}$. It is the isomorphism because every relative cohomology class contains a unique Dirichlet harmonic field.

We have thus defined all terms of diagram (7.3). Now, we have to check that the diagram is commutative.

The commutativity of the square

 $\begin{array}{ccc} \tilde{H}^k(M) & \stackrel{\tilde{i}^*}{\longrightarrow} & H^k(\partial M) \\ \lambda \downarrow & \iota \downarrow \end{array}$

 $H^k(M) \xrightarrow{i^*} H^k(\partial M)$

is almost obvious. Indeed, a form $\varphi \in \tilde{H}^k(M) = i^* \mathcal{H}^k_N(M)$ can be uniquely represented as $\varphi = i^* \omega$ with $\omega \in \mathcal{H}^k_N(M)$. Then

$$i^*\lambda\varphi = [i^*\omega]_{\partial M} = [\varphi]_{\partial M} = \tilde{i}^*\varphi.$$

Next, we check the commutativity of the square

$$ilde{H}^k(M,\partial M) \stackrel{ ilde{j}^*}{\longrightarrow} ilde{H}^k(M)$$

$$\mu \downarrow \qquad \lambda$$

 $H^k(M,\partial M) \xrightarrow{f} H^k(M)$

Let $\varphi \in \tilde{H}^k(M, \partial M) = i^* \mathcal{H}_N^{n-k}(M)$. Represent it as

$$\varphi = i^* \omega, \quad \omega \in \mathcal{H}_N^{n-k}(M).$$

⁴⁵₄₆ By the definition of μ ,

$$\mu\varphi = [\star\omega]_{(M,\partial M)}.$$

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<i>M. Belishev, V. Sharafutdinov / Bull. Sci. math.</i> $\bullet \bullet \bullet$ ($\bullet \bullet \bullet \bullet$) $\bullet \bullet \bullet - \bullet \bullet \bullet$	17
Therefore	
$j^*\mu\varphi = [\star\omega]_M.$	(7.4)
The form $\psi = \tilde{j}^* \varphi \in \tilde{H}^k(M) = i^* \mathcal{H}_N^k(M)$ can be also represented as	
$\tilde{j}^* \varphi = \psi = i^* v, v \in \mathcal{H}^k_N(M).$	
By the definition of λ ,	
$\lambda \tilde{j}^* \varphi = \lambda \psi = [\nu]_M.$	(7.5)
Comparing (7.4) and (7.5), we see that the commutativity of the square is equivequality	valent to the
$[\star\omega]_M = [\nu]_M$	
which means that the Friedrichs decomposition of the form $\star \omega$ must look as follows	5:
$\star \omega = \nu + d\alpha, \alpha \in \Omega^{k-1}(M).$	(7.6)
By the remark at the end of Section 2, we can assume the form α to satisfy the equa	tions
$\Delta \alpha = 0, \qquad \delta \alpha = 0.$	
Restricting equation (7.6) to the boundary, we have	
$\psi = i^* v = -i^* d\alpha = -di^* \alpha.$	(7.7)
On the other hand, applying \star to (7.6), we obtain	
$(-1)^{k(n-k)}\omega = \star v + \star d\alpha.$	
Take the restriction of the last equation to the boundary	
$(-1)^{k(n-k)}\varphi = (-1)^{k(n-k)}i^*\omega = i^*(\star d\alpha) = \Lambda j^*\alpha.$	
From this	
$i^*\alpha = (-1)^{k(n-k)}\Lambda^{-1}\varphi.$	
Substituting the latter value into (7.7), we obtain	
$\psi = (-1)^{kn+k+1} d\Lambda^{-1} \varphi = (-1)^{kn+k+1} T \varphi$	
or	
$\tilde{j}^*\varphi = (-1)^{kn+k+1}T\varphi.$	
This is just our definition of \tilde{j}^* . Finally, we check the commutativity of the square	
$egin{array}{ccc} H^k(\partial M) & \stackrel{ ilde{\partial}^*}{\longrightarrow} & ilde{H}^{k+1}(M,\partial M) \ \iota \downarrow & \mu \downarrow \end{array}$	
$H^k(\partial M) \xrightarrow{\partial^*} H^{k+1}(M,\partial M)$	
Given a closed form $\varphi \in \Omega^k(\partial M)$, let $\omega \in \Omega^k(M)$ be a solution to the boundary va (3.2). Then	alue problem
$\partial^* [\varphi]_{\partial M} = [d\omega]_{(M,\partial M)}$	(7.8)

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11D:BUSCI AD: 2223 /FLA [a1+; v 1.68; Prn:13/12/2006; 11:11] P.18 (1-18] 8 <i>M. Belishev, V. Sharafudinov / Bull. Sci. math.</i> ••• (••••) •••••• and $A\varphi = i^*(*d\omega)$. By the definition of μ , $\mu A\varphi = [**d\omega]_{(M,\partial M)} = (-1)^{kn+k+n+1} [d\omega]_{(M,\partial M)}$. (7.9) Comparing (7.8) and (7.9), we obtain $\mu A\varphi = (-1)^{kn+k+n+1} \partial^*[\varphi]_{\partial M}$. According to our definition of $\tilde{\partial}^*$, the last equation means that $\mu \tilde{\partial}^*[\varphi]_{\partial M} = \partial^*[\varphi]_{\partial M}$. We have thus proved the commutativity of diagram (7.3). Let us mention the following sup plement to Lemma 7.1: Corollary 7.2. If $\beta_{k-1}(\partial M) = \beta_k(\partial M) = 0$, then the Hilbert transform $T: i^*Th_M^{n-k}(M) \longrightarrow i^*H_N^k(M)$ is the isomorphism. In conclusion, we emphasize the key role of the operators A and T in the construction of sequence (7.2). Probably, such a role inscribes the DN map and Hilbert transform into the lis of objects of algebraic topology. We also set up an important open question. Recall that the schomology spaces $H^*(M) = \bigoplus_{k=0}^{H} H^k(M)$ and $H^*(M, \partial M) = \bigoplus_{k=0}^{H} H^k(M, \partial M)$ have the sturbers cannot answer the question. References 11 M.1. Belishev, The Calderon problem for two-dimensional manifolds by the BC-method, SIAM J. Math. Anal. 35 (1 (2005) 172–182. 21 M.1. Belishev, Some remarks on the impedance tomography problem for 3d-manifolds, CUBO A Math. J. 7 (1 (2005) 172–182. 21 M.1. Belishev, Some remarks on the impedance tomography problem for harmonic differential forms, Asympt Anal. 41 (2) (2005) 93–106. 31 A. Dohi, Lectures on Algebraic Topology, Springer-Verlag, Berlin, 1972. 31 A. Dohi, Lectures on Algebraic Topology, Springer-Verlag, Berlin, 1972. 31 A. Dohi, Lectures on Algebraic Topology, Springer-Verlag, Berlin, 1972. 31 A. Dohi, Lectures on Algebraic Topology, Springer-Verlag, Berlin, 1972. 31 A. Looi, Uthmann, Determining anisotropic real-analytic conductivities by boundary measurements, Comm. New App. Math. 42 (1989) 1007–1712. 31 J. Lee, G. Uthmann, Determining the Riemannian manifold from the Di		ARTICLE IN PRESS
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