Simulation of Instability of Bright Solitons for NLS with Saturating Nonlinearity ⋆

V.S.Buslaev V.E.Grikurov

Dept. Math. & Comp. Phys., Division of Physiscs, St.Petersburg Univ., 198504 Russia

Abstract

The paper deals with the generalized 1+1 nonlinear Schrödinger equation

$$iu_t + u_{xx} + \mid u \mid^{2p} u - \alpha \mid u \mid^{2q} u = 0, \quad \alpha > 0, \quad q > p,$$

which is the model of laser propagation throughout nonlinear optic materials with a saturation. We are focus on the numerical study of the effect of soliton's "self-compression" under small perturbations and on the trapping phenomena due to collisions.

Key words: nonlinear Schrödinger equation, non-Kerr's nonlinearity, solitons, instability, collisions

1 Description of the problem and motivations of the model

The nonlinear Schrödinger (NLS) equation

$$iu_t = -\nabla^2 u + F\left(\mid u \mid^2\right) u \tag{1.1}$$

⋆ The work was made possible in part due to INTAS grant IR-97-0812.
Email addresses: buslaev@mph.phys.spbu.ru (V.S.Buslaev), grikurov@mph.phys.spbu.ru (V.E.Grikurov).

arises as the description of the lasers propagation through nonlinear optic materials and Langmuir waves in a plasma. Under certain (nonrestrictive) conditions on the nonlinearity $F(\xi)$ [7] equation (1.1) admits a one-parametric family of localized, finite energy traveling wave solutions — solitary waves or, simply, solitons. These solutions are of special importance due to the role they play in the large-time asymptotics of the initial value problem.

In what follows we consider equation (1.1) in one space dimension and assume that the nonlinearity $F(\xi)$ be

$$F(\xi) = -2(p+1)\xi^p + \alpha(q+1)\xi^q, \quad \alpha > 0, \quad q > p \tag{1.2}$$

(coefficients are normalized for some convenience). This model approximates the saturating law $F(\xi) = -\text{const}\xi^p/(1+\gamma\xi^p)$.

The bright soliton traveling with the speed v is the solution of (1.1)– (1.2) of the form $u_s(x-vt;\omega) = \exp(i(\omega - v^2/4)t + ivx/2)u_0(x-vt;\omega)$, where ω is the parameter and the soliton's envelop $u_0(z;\omega)$ may be expressed in quadrature. If $q = 2p$, this quadrature is reduced to the explicit form:

$$u_o(z;\omega) = \left[\omega \,/\, \left(1 + \sqrt{1-\alpha\omega}\cosh(2p\sqrt{\omega}z)\right)\right]^{1/2p} \tag{1.3}$$

First, we are interested in the stability properties of the solution $u_s(x-vt;\omega)$. The criterion is well known: the inequality

$$N'(\omega) > 0\,, \quad N(\omega) = \int_{-\infty}^{+\infty} u_0^2(x;\omega)dx \tag{1.4}$$

yields the stability (see, e.g., [1] and references therein for review; the analysis of the scattering properties of a stable soliton is given in [2]).

As an example, for the power law nonlinearity $F(\xi) = -\xi^p$ the stability criterion in d-dimensions is $pd < 2$. If $pd \geqslant 2$ (so called supper-critical case) then the typical behaviour of the solution is blow-up, or collapse, i.e. an infinite growth of the amplitude for a finite time. From the another hand, if the function $F(\xi)$ is not too rapidly goes to minus infinity for large ξ ($F(\xi) \geq -C\xi^p$ as $\xi \to \infty$, $p < 2/d$ in d-dimensions), then any solution with smooth initial data is known to be a global one. However, the stability condition (1.4) may nevertheless be not valid for some ω-s, that is, the function $N(\omega)$ may have a minimum at some point ω_* (the so-called instability threshold). By another words, the equation may admit the existence of both unstable and stable solitons. So the question is: what is the large-time behaviour of the unstable ($\omega < \omega_*$) soliton under small perturbation? *

* The instability threshold arises for the model (1.2) in one-dimension if $p \geqslant 3$. Such nonlinearity looks rather exotic and is utilized here as the simplest model.

Second, we examine collisions of solitons of sufficiently large amplitude. It can results in different scenarios. However, it is possible to specify the properties of the initial data which lead to solution with a certain structure at a large-time. Normally, the trapping of energy into few channels is observed. The kind of trapping and the post-trapped behavior depend on the energy of colliding solitons.

The paper mostly concerns numerical experiments which are related to phenomena in question. Simulation of the propagation of an unstable soliton is discussed in Section 2. This Section also contains references to Appendices A–C the we discuss the application of the perturbation technique to the analysis of soliton's stability. Section 3 contains observations of trapping phenomena arising due to collisions of solitons.

2 Self-compression and multi-focusing

In this section we deal with the ground ($v = 0$) soliton.

Due to numerics (see Fig. 2.1), a perturbation of an unstable soliton generally displays the following typical scenario of "self-compression". After some time of "stable" propagation (the only this time depends on a perturbation) the solution abruptly transforms to a new higher amplitude level (which is only weakly depends on a perturbation); the process of transformation is concomitant by the formation of a rapid radiation; the post-transformed behaviour is the relaxing oscillation of the amplitude, each period of oscillation also gives rise to some radiation (these oscillations are called "multi-focusing", see also [3]); the large-time solution is close to another soliton corresponding to a parameter $\omega_\infty > \omega_*$. The scale and the strength of post-transformed oscillations depend on the coefficient α in (1.2): smaller α-s correspond to higher oscillations (see also Fig. 2.2) .

The results which are shown in Fig 2.1 and Fig. 2.2 correspond to the initial data $U_\omega(x,t)|_{t=0}$, where function $U_\omega(x,t)$ is the special perturbation series around the initial soliton (see formula (A.9) in Appendix A). However, this function is a somehow universal initial data to provide the given scenario since any perturbation (e.g., a numerical noise) with a non-zero projection on the eigenstate $\varphi(x;\omega)$ (A.7) of the linearized problem develops into self-compression and multi-focusing. Scale ε of a perturbation is equivalent to a shift in the time.

In the left-hand part of Fig. 2.2 one can see the behavior of solutions cor-

Note that in higher dimensions restriction on p is quite natural.

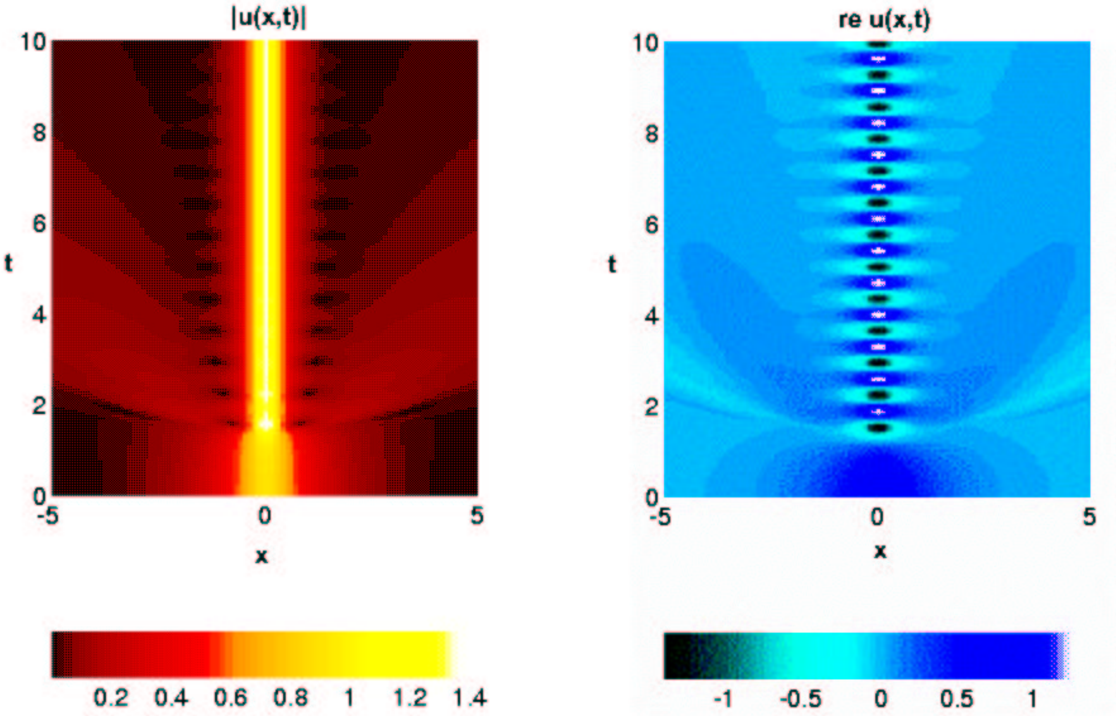

Fig. 2.1. Numerical simulation of the initial-value problem for the model (1.1)–(1.2) with $p = 3$, $q = 6$, $\alpha = 0.1$: initial data are the sum of first two items of the series (A.9) computed with $\omega = 1$, $\varepsilon = 0.1$. *Left:* module of the solution; *Right:* real part of the solution.

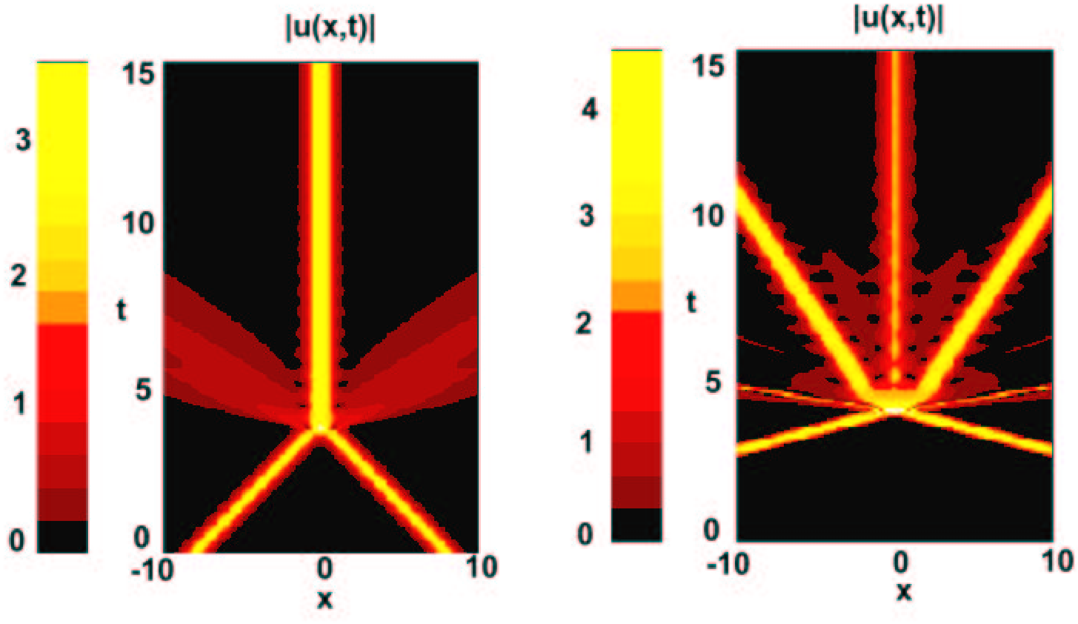

Fig. 3.1. Numerical simulation (the module of the solution is shown) of the initial-value problem for the model (1.1)–(1.2), (3.1) with $p = 1$, $q = 2$ and $\alpha = 0.1$. *Left:* the parameters ω and v in (3.1) are such that $\mathcal{H}$ is large negative; *Right:* the parameters ω and v in (3.1) are such that $\mathcal{H}$ is large positive.

responding to small α-s. The important feature of the obtained numerical results is as follows: post-transformed solutions obey the scaling property $u \rightsquigarrow \alpha^{-2(q-p)} u\left(t/\alpha, x/\sqrt{\alpha}\right)$ while the initial data of these solutions do not.

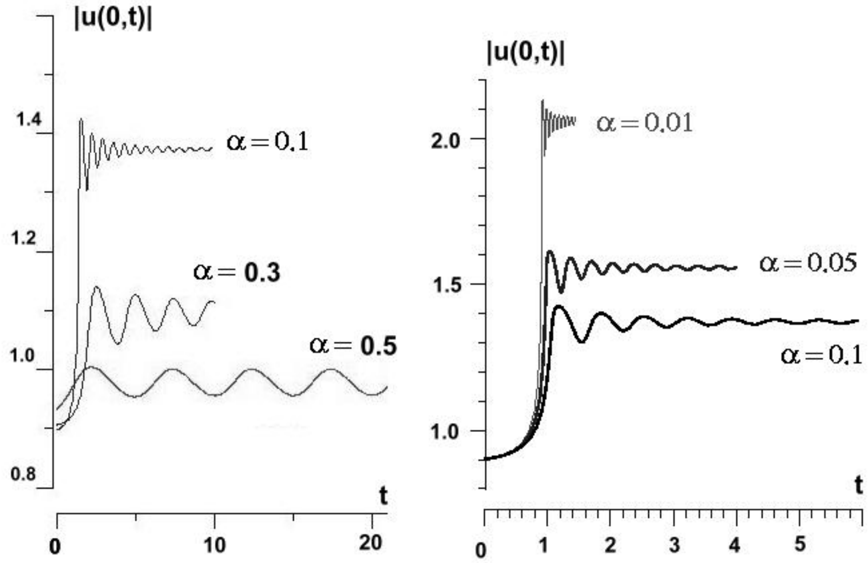

Fig. 2.2. Numerical simulation of the initial-value problem for the model (1.1)–(1.2) with $p = 3$, $q = 6$: module of the solution depending on t at $x = 0$; initial data are the sum of first two items of the series (A.9) computed with $\omega = 1$, $\varepsilon = 0.1$ (instability threshold arises at this ω for $\alpha_* \sim 0.57\dots$). *Left:* solutions corresponding to values of α which are close to threshold ones; *Right:* solutions corresponding to small values of α.

The special care should be paid to the case ω is close to ω_*, where ω corresponds to the initial soliton. It is seen in the left-hand part of Fig. 2.2 that the solution corresponding to α which is close to the instability threshold α_* (for the given ω) exhibits the long-live periodic oscillations. This behavior is in agreement with the adiabatic solution which is described in Appendix B. Radiation loses of this solution are small (in fact — exponentially small, see [4]).

The post-transformed oscillations are very likely caused by the excitation of an internal mode corresponding to a complex eigenvalue inside the gap of continuous spectrum of the linearized problem. Such modes have been studied in [2], [6]. However, the mechanism of the excitation of this modes by the initial data is still an open question.

3 Trapping of colliding solitons

In this section we discuss the simulation results for the cubic-quintic model (1.1)–(1.2) (that is, with $p = 1$ and $q = 2$ which is contrary to $p = 3$ and $q = 6$ used in the previous section). The initial data be

$$u\big|_{t=0} = u_0(x + x_0; \omega)e^{ivx/2} + u_0(x - x_0; \omega)e^{-ivx/2} \tag{3.1}$$

(to be specific, we examine only even solutions). Generally, (3.1) is two-parameter (ω and v; x_0 is not essential for the discussion) function. However, the classification of results may be done by means of the single parameter

$$\mathcal{H} = \int_{-\infty}^{+\infty} \left[\left| \frac{\partial u}{\partial x} \right|^2 + \int\limits_0^{|u|^2} F(\xi) d\xi \right] dx \Bigg|_{t=0} \tag{3.2}$$

(the value $\mathcal{H}$ is known to be the conserving quantity of the equation (1.1)).

The results of simulation are shown in Fig. 3.1–Fig. 3.2. One can observe the trapping of colliding solitons and the distribution of its energy into few channels. The propagation inside each of these channels exhibits the soliton-like behaviour with, possibly, modulated parameters. The process of trapping burns the dispersive radiation. The mode of trapping depends on the sign of $\mathcal{H}$: the left-hand part in Fig. 3.1 corresponds to negative $\mathcal{H}$ with the typical trapping "two to one" while for positive $\mathcal{H}$ the situation may be much more complicated (see the right-hand part in Fig. 3.1).

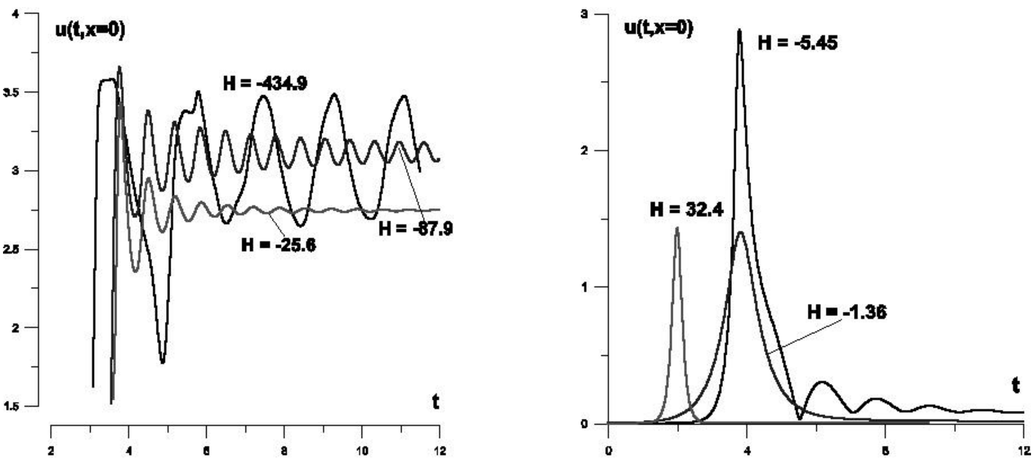

Fig. 3.2. Module of the post-trapped solution depending on t at $x = 0$; model (1.1)–(1.2), (3.1) with $p = 1$, $q = 2$ and $\alpha = 0.1$. *Left:* solutions with large negative $\mathcal{H}$; *Right:* solutions with moderate values of $\mathcal{H}$.

Let us consider in more details the trapping of solitons with negative $\mathcal{H}$. Small values of $\mathcal{H}$ (both negative and positive) provide no trapping (see the right-hand part in Fig.3.2) and the solution decays in time. While $\mathcal{H} \to -\infty$ the trapping begins to appear (the left-hand part in Fig.3.2). For not too large values of $-\mathcal{H}$ the trapped solution after some relaxing time looks like a new ground soliton. However, the following increasing of $-\mathcal{H}$ yields at a large time the solution oscillating in time (each oscillation also give rise to a dispersive radiation).

It is also worth to note the similarity between large-time behavior of solutions which are described in the previous and in the given sections.

A Construction of the perturbation series

Consider the linearization of the complexified equation (1.1) around the soliton (1.3):

$$u(x,t;\omega) = e^{i\omega t}\left[u_0(x;\omega) + \varepsilon u_1(x,t;\omega) + \dots\right], \tag{A.1}$$

where ε is a formally introduced parameter. Consequent approximations $u_n(t,x) = u_{n1}(t,x) + iu_{n2}(t,x)$ satisfy the recurrent equations

$$\partial_t \begin{pmatrix} u_{n1} \\ u_{n2} \end{pmatrix} = \mathbf{H}(x;\alpha) \begin{pmatrix} u_{n1} \\ u_{n2} \end{pmatrix} + \mathbf{M}_n\left(u_0, u_1, \dots, u_{n-1}\right) \tag{A.2}$$

with the operator

$$\mathbf{H}(\omega) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(-\partial_x^2 + \omega\right) + \begin{pmatrix} 0 & , F(u_0^2) \\ -F(u_0^2) - 2F'(u_0^2)u_0^2 \ , & 0 \end{pmatrix}, \tag{A.3}$$

and the right-hand sides

$$\mathbf{M}_n\left(u_0, u_1, \dots, u_{n-1}\right) = \textstyle\sum_{k=1}^{n-1} F_{n-k} \begin{pmatrix} 0 & , 1 \\ -1 & , 0 \end{pmatrix} \begin{pmatrix} u_{k1} \\ u_{k2} \end{pmatrix} + G_{n-1} \begin{pmatrix} 0 \\ -1 \end{pmatrix} u_0\,, \tag{A.4}$$

where

$$F_k(u_0, u_1, \dots, u_k) = \tfrac{1}{k!}\tfrac{d^k}{d\varepsilon^k} F\left(|u_0 + \varepsilon u_1 + \dots|^2\right)\Big|_{\varepsilon=0}\,, \tag{A.5}$$

$$G_{n-1}(u_0, u_1, \dots, u_{n-1}) = F_n(u_0, u_1, \dots, u_n) - F'(u_0^2)u_0(u_n + \bar{u}_n) \tag{A.6}$$

The functions (A.5) and (A.6) are evidently real and hold the following property: $F_k(u_0, \gamma u_1, \dots, \gamma^k u_k) = \gamma^k F_k(u_0, u_1, \dots, u_k)$, same for G_{n-1}.

Equation (A.2) is uniform when $n = 1$, i.e., $\mathbf{M_1} \equiv 0$. It is known that if $\omega < \omega_*$ (i.e., if an unstable soliton is perturbed) the operator (A.3) has real point spectrum (see numerical examples in Appendix C). This implies the following way to fix the series (A.1). Let $\lambda(\omega)$ be the largest positive eigenvalue of the operator (A.3) and $\varphi(x;\omega)$ be the corresponding eigenfunction:

$$\mathbf{H}(\omega)\varphi(x;\omega) = \lambda(\omega)\varphi(x;\omega)\,. \tag{A.7}$$

Then the functions $u_n(t,x;\omega) = \exp(n\lambda(\omega)t)\psi_n(x;\omega)$, where $\psi_1(x;\omega) = \varphi(x;\omega)$ and

$$\left[\mathbf{H}(\omega) - n\lambda(\omega)\right]\psi_n = \mathbf{M}_n\left(u_0, \psi_1, \dots, \psi_{n-1}\right)\,, \quad n > 1, \tag{A.8}$$

satisfy the recurrent equation (A.2). Due to the choice of $\lambda(\omega)$ the operator in (A.8) is invertible for any n. Finally, the perturbation series (A.1) reads

$$U_\omega(x,t) = u_0(x;\omega) + \textstyle\sum_{n=1}^{\infty} z^n\psi_n(x;\omega)\big|_{z=\varepsilon e^{\lambda(\alpha)t}}\,. \tag{A.9}$$

It may be shown by applying of the WKB technique for large n to equation (A.8) that functions $\psi_n(x;\omega)$ are uniformly bounded. Thus the series (A.9) converges in some non-zero circle of z-plane. However, the convergence becomes slow when $\lambda(\omega) \to 0$ which corresponds to $\omega \to \omega_*$. This case the another asymptotic *ansatz* should be applied and it is discussed in Appendix B.

B Adiabatic solution in the neighborhood of the threshold

We discuss here the adiabatic approximation of the solution which is close to $u_s(x,t;\omega_*)$. Our construction has common features with the approach of the paper [5].

Let the solution be sought in the form of the following asymptotic series:

$$u(x,t) = \exp\left(\frac{i}{\epsilon}\int\omega(\tau)d\tau\right)\sum_{j\geqslant 0} v_j(x,\tau)\epsilon^j\,, \quad \omega(\tau;\epsilon) = \sum_{j\geqslant 0}\omega_j(x,\tau)\epsilon^j\,, \quad \tau = \epsilon t\,. \tag{B.1}$$

Substitution of (B.1) into equation (1.1) yields in increasing orders of ϵ:

$$\begin{aligned} &\epsilon^0 : v_0(x;\tau) = u_0(x;\omega(\tau))\,; && \epsilon^1 : \omega_0 = \text{const}\,, \omega_1 = 0\,, u_1 = 0\,; \\ &\epsilon^2 : v_2(x;\tau) = \omega_2(\tau)\left.\frac{\partial u_0}{\partial\omega}\right|_{\omega_0}\,; && \epsilon^3 : v_2(x;\tau) = -i\omega_2(\tau)'\hat{u}_0(x;\omega_0)\,, \end{aligned} \tag{B.2}$$

where the function $\hat{u}_0(x;\omega_0)$ be the solution of the equation

$$\left(-\partial_x^2 + \omega_0 + F(u_0^2)\right)\hat{u}_0 = \left.\frac{\partial u_0}{\partial\omega}\right|_{\omega_0}\,. \tag{B.3}$$

The solvability condition for this equation implies the orthogonality of $\left.\frac{\partial u_0}{\partial \omega}\right|_{\omega_0}$ and of $u_0|_{\omega_0}$ and, consequently, $\omega_0 = \omega_*$. Moreover, in orders of ϵ^4 and ϵ^5 one obtains equations to define functions $v_4(x;\tau)$ and $\omega_2(\tau)$. In particular,

$$M_* \omega_2'' + \frac{1}{2} N''(\omega_*) \omega_2^2 = \text{const}\,, \quad M_* = 2 \int \left(\frac{1}{u_s} \int_0^x u_s \frac{\partial u_s}{\partial \omega} dx' \right)^2 dx\,. \qquad \text{(B.4)}$$

The equation (B.4) admits periodic solutions. In accordance with (B.2) a periodic modulation of the phase is also concomitant by a periodic modulation of the amplitude which is in accordance with dependencies of the left-hand part in Fig.2.2.

C The real spectrum of the operator H

Note that the equation (1.1)–(1.2) is invariant under the scaling transformation

$$\omega t \rightsquigarrow t, \quad \sqrt{\omega} x \rightsquigarrow x, \quad \omega^{1/2p} u \rightsquigarrow u, \quad \omega^\gamma \alpha \rightsquigarrow \alpha\,, \qquad \text{(C.1)}$$

where $\gamma = q/p - 1$. Apparently, the specification of the coefficients in (1.2) is not essential and also may be removed by an appropriate scaling transformation. So the parameter ω may be fixed without loss of generality of the model (1.1)–(1.2), and the following tabulation is of universal character.

Table C.1 summarizes the numerical estimate of real non-zero eigenvalues for different α-s provided $\omega = 1$; only one (up to reflection with respect to the origin) such eigenvalue is observed for each α.

α	$\pm\lambda(\alpha)$	α	$\pm\lambda(\alpha)$
0.00	2.905088288178651	0.20	2.278448447143798
0.01	2.876873842958459	0.30	1.901772557261468
0.02	2.848373136366166	0.40	1.459247643166726
0.05	2.761094928083689	0.50	0.893960887951942
0.10	2.609311186479360	0.55	0.474338804126698

Table C.1
Real point spectrum (different from zero) of the operator (A.3); model (1.2) with $p = 3$, $q = 6$, $\omega = 1$ (instability threshold arises at this ω for $\alpha_* \sim 0.57\ldots$).

References

[1] J.L.Bona and A.Soyer, On the stability of solitory-wave solutions of model equations for long waves *J.Nonlinear Sci.* **4** (1994) 449–470.

[2] V.S.Buslaev and G.S.Perelman, Scattering for the nonlinear Schrödinger equation: states that are close to a soliton *Algebra and Analysis* **4** (1992) 63–102.

[3] B.J.LeMesurier, G.Papanicolaou, C.Sulem and P.L.Sulem. Focusing and multi-focusing solutions of the nonlinear Schrödinger equation *Physica D* **31** (1988) 78–102.

[4] D.E.Pelinovsky. Radiative effects to the adiabatic dynamics of envelope-wave solitons. *Physica D* **119** (1998) 301–313.

[5] D.E.Pelinovsky, V.V.Afanasjev, and Yu.S.Kivshar. Nonlinear theory of oscillating, decaying, and collapsing solitons in the generalized nonlinear Schrödinger equation *Rhysical Review* **E-53** (1996) 1940–1953.

[6] D.E.Pelinovsky, V.V.Afanasjev, and Yu.S.Kivshar. Internal modes of envelope solitons. *Physica D* **116** (1998) 121–142.

[7] W.A.Strauss. Existence of solitory waves in higher dimensions *Comm.Math.Phys.* **55** (1977) 149–162.